

Heat Kernels for Non-symmetric Non-local Operators

Zhen-Qing Chen^{*} and Xicheng Zhang[†]

Abstract

We survey the recent progress in the study of heat kernels for a class of non-symmetric non-local operators. We focus on the existence and sharp two-sided estimates of the heat kernels and their connection to jump diffusions.

AMS 2000 Mathematics Subject Classification: Primary 60J35, 47G30, 60J45; Secondary: 31C05, 31C25, 60J75

Keywords: Discontinuous Markov process, diffusion with jumps, non-local operator, pseudo-differential operator, heat kernel estimate, Lévy system

1 Introduction

Second order elliptic differential operators and diffusion processes take up, respectively, an central place in the theory of partial differential equations (PDE) and the theory of probability. There are close relationships between these two subjects. For a large class of second order elliptic differential operators \mathcal{L} on \mathbb{R}^d , there is a diffusion process X on \mathbb{R}^d associated with it so that \mathcal{L} is the infinitesimal generator of X , and vice versa. The connection between \mathcal{L} and X can also be seen as follows. The fundamental solution (also called heat kernel) for \mathcal{L} is the transition density function of X . For example, when

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i},$$

where $(a_{ij}(x))_{1 \leq i,j \leq d}$ is a $d \times d$ symmetric matrix-valued continuous function on \mathbb{R}^d that is uniformly elliptic and bounded, and $b(x) = (b_1(x), \dots, b_d(x))$ is a bounded \mathbb{R}^d -valued function, there is a unique diffusion $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ on \mathbb{R}^d that solves the martingale problem for $(\mathcal{L}, C_c^2(\mathbb{R}^d))$. That is, for every $x \in \mathbb{R}^d$, there is a unique probability measure \mathbb{P}_x on the space $C([0, \infty); \mathbb{R}^d)$ of continuous functions on \mathbb{R}^d so that $\mathbb{P}_x(X_0 = x) = 1$ and for every $f \in C_c^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s) ds$$

^{*}Research partially supported by NSF Grant DMS-1206276.

[†]Research partially supported by NNSFC grant of China (Nos. 11271294, 11325105).

is a \mathbb{P}_x -martingale. Here $X_t(\omega) = \omega(t)$ is the coordinate map on $C([0, \infty); \mathbb{R}^d)$. It is also known that (X, \mathbb{P}_x) is the unique weak solution to the following stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x,$$

where W_t is an n -dimensional Brownian motion and $\sigma(x) = a(x)^{1/2}$ is the symmetric square root matrix of $a(x) = (a_{ij}(x))_{1 \leq i, j \leq d}$.

When a is Hölder continuous, it is known that \mathcal{L} has a jointly continuous heat kernel $p(t, x, y)$ with respect to the Lebesgue measure on \mathbb{R}^d that enjoys the following Aronson's estimate (see Theorem 4.3 below): there are constants $c_k > 0$, $k = 1, \dots, 4$, so that

$$c_1 t^{-d/2} \exp(-c_2 |x - y|^2/t) \leq p(t, x, y) \leq c_3 t^{-d/2} \exp(-c_4 |x - y|^2/t) \quad (1.1)$$

for $t > 0$ and $x, y \in \mathbb{R}^d$.

As many physical and economic systems exhibit discontinuity or jumps, in-depth study on non-Gaussian jump processes are called for. See for example, [5, 29, 34, 40] and the references therein. The infinitesimal generator of a discontinuous Markov process in \mathbb{R}^d is no longer a differential operator but rather a non-local (or, integro-differential) operator. For instance, the infinitesimal generator of an isotropically symmetric α -stable process in \mathbb{R}^d with $\alpha \in (0, 2)$ is a fractional Laplacian operator $c\Delta^{\alpha/2} := -c(-\Delta)^{\alpha/2}$. During the past several years there is also many interest from the theory of PDE (such as singular obstacle problems) to study non-local operators; see, for example, [8, 42] and the references therein. Quite many progress has been made in the last fifteen years on the development of the De Giorgi-Nash-Moser-Aronson type theory for non-local operators. For example, Kolokoltsov [35] obtained two-sided heat kernel estimates for certain stable-like processes in \mathbb{R}^d , whose infinitesimal generators are a class of pseudo-differential operators having smooth symbols. Bass and Levin [3] used a completely different approach to obtain similar estimates for discrete time Markov chain on \mathbb{Z}^d , where the conductance between x and y is comparable to $|x - y|^{-n-\alpha}$ for $\alpha \in (0, 2)$. In [17], two-sided heat kernel estimates and a scale-invariant parabolic Harnack inequality (PHI in abbreviation) for symmetric α -stable-like processes on d -sets are obtained. Recently in [18], two-sided heat kernel estimates and PHI are established for symmetric non-local operators of variable order. The De Giorgi-Nash-Moser-Aronson type theory is studied very recently in [19] for symmetric diffusions with jumps. We refer the reader to the survey articles [10, 28] and the references therein on the study of heat kernels for symmetric non-local operators. However, for non-symmetric non-local operators, much less is known. In this article, we will survey the recent development in the study of heat kernels for non-symmetric non-local operators. We will concentrate on the recent progress made in [25, 26] and [13]. In Section 5 of this paper, we summarize some other recent work on heat kernels for non-symmetric non-local operators. We also take this opportunity to fill a gap in the proof of [25, (3.20)], which is (3.23) of this paper. The proof in [25] works for the case $|x| \geq t^{1/\alpha}$. In Section 3, a proof is supplied for the case $|x| \leq t^{1/\alpha}$. In fact, a slight modification of the original proof for [25, Theorem 2.5] gives a better estimate (3.20) than (3.23).

In this survey, we concentrate on heat kernel on the whole Euclidean spaces and on the work that the authors are involved. We will not discuss Dirichlet heat kernels in this article.

2 Lévy process

A Lévy process on \mathbb{R}^d is a right continuous process $X = \{X_t; t \geq 0\}$ having left limit that has independent stationary increments. It is uniquely characterized by its Lévy exponent ψ :

$$\mathbb{E}_0 \exp(i\xi \cdot X_t) = \exp(-t\psi(\xi)), \quad \xi \in \mathbb{R}^d. \quad (2.1)$$

Here for $x \in \mathbb{R}^d$, the subscript x in the mathematical expectation \mathbb{E}_x and the probability \mathbb{P}_x means that the process X_t starts from x . The Lévy exponent ψ admits a unique decomposition:

$$\psi(\xi) = ib \cdot \xi + \sum_{i,j=1}^d a_{ij} \xi_i \xi_j + \int_{\mathbb{R}^d} \left(1 - e^{i\xi \cdot z} + i\xi \cdot z \mathbb{1}_{\{|z| \leq 1\}}\right) \Pi(dz), \quad (2.2)$$

where $b \in \mathbb{R}^d$ is a constant vector, (a_{ij}) is a non-negative definite symmetric constant matrix, and $\Pi(dz)$ is a positive measure on $\mathbb{R}^d \setminus \{0\}$ so that $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \Pi(dz) < \infty$. The Lévy measure $\Pi(dz)$ has a strong probabilistic meaning. It describes the jumping intensity of X making a jump of size z . Denote by $\{P_t; t \geq 0\}$ the transition semigroup of X ; that is, $P_t f(x) = \mathbb{E}_x f(X_t) = \mathbb{E}_0 f(x + X_t)$. For an integrable function f , its Fourier transform is defined to be $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) dx$. Then we have by (2.1) and Fubini's theorem,

$$\widehat{P_t f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mathbb{E}_0 f(x + X_t) dx = \mathbb{E}_0 \left[e^{-\xi \cdot X_t} \left(\int_{\mathbb{R}^d} e^{i\xi \cdot (x + X_t)} f(x + X_t) dx \right) \right] = e^{-t\psi(-\xi)} \widehat{f}(\xi).$$

If we denote the infinitesimal generator of $\{P_t; t \geq 0\}$ (or X) by \mathcal{L} , then

$$\widehat{\mathcal{L}f}(\xi) = \frac{d}{dt} \Big|_{t=0} \widehat{P_t f}(\xi) = -\psi(-\xi) \widehat{f}(\xi). \quad (2.3)$$

Hence $-\psi(-\xi)$ is the Fourier multiplier (or symbol) for the infinitesimal generator \mathcal{L} of X . One can derive a more explicit expression for the generator \mathcal{L} : for $f \in C_c^2(\mathbb{R}^d)$,

$$\mathcal{L}f(x) = \sum_{i,j=1}^d a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + b \cdot \nabla f(x) + \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}}) \Pi(dz). \quad (2.4)$$

When $b = 0$, $\Pi = 0$ and $(a_{ij}) = \mathbf{I}_{d \times d}$ the identity matrix, that is when $\psi(\xi) = |\xi|^2$, X is a Brownian motion in \mathbb{R}^d with variance $2t$ and infinitesimal generator $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. When $b = 0$, $a_{ij} = 0$ for all $1 \leq i, j \leq d$ and $\Pi(dz) = \mathcal{A}(d, -\alpha) |z|^{-(d+\alpha)} dz$ for $0 < \alpha < 2$, where $\mathcal{A}(d, -\alpha)$ is a normalizing constant so that $\psi(\xi) = |\xi|^\alpha$, X is a rotationally symmetric α -stable process in \mathbb{R}^d , whose infinitesimal generator is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$.

Unlike Brownian motion case, explicit formula for the transition density function of symmetric α -stable processes is not known except for a very few cases. However we can get its two-sided estimates as follows. It follows from (2.1) that under \mathbb{P}_0 , (i) AX_t has the same distribution as X_t

for every $t > 0$ and rotation A (an orthogonal matrix); (ii) for every $\lambda > 0$, $X_{\lambda t}$ has the same distribution as $\lambda^{1/\alpha} X_t$. Let $p(t, x)$ be the density function of X_t under \mathbb{P}_0 ; that is,

$$p(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi.$$

Then $p(t, x)$ is a function of t and $|x|$ and $p(t, x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x)$. Using Fourier's inversion, one gets

$$\lim_{|x| \rightarrow \infty} |x|^{d+\alpha} p(1, x) = \alpha 2^{\alpha-1} \pi^{-(d/2+1)} \sin(\alpha\pi/2) \Gamma((d+\alpha)/2) \Gamma(\alpha/2).$$

(See Pólya [39] when $d = 1$ and Blumenthal-Gettoor [6, Theorem 2.1] when $d \geq 2$.) It follows that $p(1, x) \asymp 1 \wedge \frac{1}{|x|^{d+\alpha}}$. Consequently,

$$p(t, x) \asymp t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \asymp \frac{t}{(t^{1/\alpha} + |x|)^{d+\alpha}}. \quad (2.5)$$

Here for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$, and for two functions f, g , $f \asymp g$ means that f/g is bounded between two positive constants.

In real world, almost every media we encounter has impurities so we need to consider state-dependent stochastic processes and state-dependent local and non-local operators. Intuitively speaking, we need to consider processes and operators where $\psi(\xi)$ is dependent on x ; that is, $\psi(x, \xi)$. If one uses Fourier multiplier approach (2.3), one gets pseudo differential operators. The connection between pseudo differential operators and Markov processes has been nicely exposted in N. Jacob [30]. In this survey, we take (2.4) as a starting point but with $a_{ij}(x)$, $b(x)$ and $\Pi(x, dz)$ being functions of $x \in \mathbb{R}^d$. That is,

$$\mathcal{L}f(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}}) \Pi(x, dz).$$

We will concentrate on the case where $\Pi(x, dz) = \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz$ for some $\alpha \in (0, 2)$ and a measurable function $\kappa(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying for any $x, y, z \in \mathbb{R}^d$,

$$0 < \kappa_0 \leq k(x, z) \leq \kappa_1 < \infty, \quad \kappa(x, z) = \kappa(x, -z), \quad (2.6)$$

and for some $\beta \in (0, 1)$,

$$|\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^\beta. \quad (2.7)$$

3 Stable-like processes and their heat kernels

In this section, we consider the case where $a_{ij} = 0$, $b = 0$ and $\Pi(x, dz) = \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz$; that is,

$$\mathcal{L}f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz, \quad (3.1)$$

where $\kappa(x, z)$ is a function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (2.6) and (2.7). Here p.v. stands for the Cauchy principal value, that is,

$$\mathcal{L}f(x) = \lim_{\varepsilon \rightarrow 0} \int_{\{z \in \mathbb{R}^d: |x| \geq \varepsilon\}} (f(x+z) - f(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz.$$

Since $\kappa(x, z)$ is symmetric in z , when $f \in C_b^2(\mathbb{R}^d)$, we can rewrite $\mathcal{L}f(x)$ as

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}}) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz. \quad (3.2)$$

The non-local operator \mathcal{L} of (3.1) typically is not symmetric, as oppose to non-local operator given by

$$\tilde{\mathcal{L}}f(x) := \lim_{\varepsilon \rightarrow 0} \int_{\{y \in \mathbb{R}^d: |y-x| \geq \varepsilon\}} (f(y) - f(x)) \frac{c(x, y)}{|x-y|^{d+\alpha}} dz \quad (3.3)$$

in the distributional sense. Here $c(x, y)$ is a symmetric function that is bounded between two positive constants. The operator $\tilde{\mathcal{L}}$ is the infinitesimal generator of the symmetric α -stable-like process studied in Chen and Kumagai [17], where it is shown that $\tilde{\mathcal{L}}$ has a jointly Hölder continuous heat kernel that admits two-sided estimates in the same form as (2.5).

The following result is recently established in [25].

Theorem 3.1 ([25, Theorem 1.1]) *Under (2.6) and (2.7), there exists a unique nonnegative jointly continuous function $p(t, x, y)$ in $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving*

$$\frac{\partial}{\partial t} p(t, x, y) = \mathcal{L}p(t, \cdot, y)(x), \quad x \neq y, \quad (3.4)$$

and enjoying the following four properties:

(i) (Upper bound) *There is a constant $c_1 > 0$ so that for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,*

$$p(t, x, y) \leq c_1 t(t^{1/\alpha} + |x - y|)^{-d-\alpha}. \quad (3.5)$$

(ii) (Hölder's estimate) *For every $\gamma \in (0, \alpha \wedge 1)$, there is a constant $c_2 > 0$ so that for every $t \in (0, 1]$ and $x, y, z \in \mathbb{R}^d$,*

$$|p(t, x, z) - p(t, y, z)| \leq c_2 |x - y|^\gamma t^{1-(\gamma/\alpha)} \left(t^{1/\alpha} + |x - z| \wedge |y - z| \right)^{-d-\alpha}. \quad (3.6)$$

(iii) (Fractional derivative estimate) *For all $x \neq y \in \mathbb{R}^d$, the mapping $t \mapsto \mathcal{L}p(t, \cdot, y)(x)$ is continuous on $(0, 1]$, and*

$$|\mathcal{L}p(t, \cdot, y)(x)| \leq c_3 (t^{1/\alpha} + |x - y|)^{-d-\alpha}. \quad (3.7)$$

(iv) (Continuity) *For any bounded and uniformly continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,*

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p(t, x, y) f(y) dy - f(x) \right| = 0. \quad (3.8)$$

Moreover, we have the following conclusions.

(v) The constants c_1 , c_2 and c_3 in (i)-(iii) above can be chosen so that they depend only on $(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2)$, $(d, \alpha, \beta, \gamma, \kappa_0, \kappa_1, \kappa_2)$, and $(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2)$, respectively.

(vi) (Conservativeness) For all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$, $p(t, x, y) \geq 0$ and

$$\int_{\mathbb{R}^d} p(t, x, y) dy = 1. \quad (3.9)$$

(vii) (C-K equation) For all $s, t \in (0, 1]$ with $t + s \in (0, 1]$ and $x, y \in \mathbb{R}^d$, the following Chapman-Kolmogorov equation holds:

$$\int_{\mathbb{R}^d} p(t, x, z) p(s, z, y) dz = p(t + s, x, y). \quad (3.10)$$

(viii) (Lower bound) There exists $c_4 = c_4(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2) > 0$ so that for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$p(t, x, y) \geq c_4 t(t^{1/\alpha} + |x - y|)^{-d-\alpha}. \quad (3.11)$$

(ix) (Gradient estimate) For $\alpha \in [1, 2)$, there exists $c_5 = c_5(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2) > 0$ so that for all $x \neq y$ in \mathbb{R}^d and $t \in (0, 1]$,

$$|\nabla_x \log p(t, x, y)| \leq c_5 t^{-1/\alpha}. \quad (3.12)$$

3.1 Approach

We now sketch the main idea behind the proof of Theorem 3.1.

To emphasize the dependence of \mathcal{L} in (3.1) on κ , we write it as \mathcal{L}^κ . For each fixed $y \in \mathbb{R}^d$, we consider Lévy process (starting from 0) with Lévy measure $\Pi_y(dz) = \frac{\kappa(y, z)}{|z|^{d+\alpha}} dz$, and denote its marginal probability density function and infinitesimal generator by $p_y(t, x)$ and $\mathcal{L}^{\kappa(y)}$, respectively. Then we have

$$\frac{\partial}{\partial t} p_y(t, x) = \mathcal{L}^{\kappa(y)} p_y(t, x). \quad (3.13)$$

We use Levi's idea and search for heat kernel $p(t, x, y)$ for \mathcal{L}^κ with the following form:

$$p(t, x, y) = p_y(t, x - y) + \int_0^t \int_{\mathbb{R}^d} p_z(t - s, x - z) q(s, z, y) dz dy \quad (3.14)$$

with function $q(s, z, y)$ be determined below. We want

$$\frac{\partial}{\partial t} p(t, x, y) = \mathcal{L}^\kappa p(t, \cdot, y)(x) = \mathcal{L}^{\kappa(x)} p(t, \cdot, y)(x).$$

Formally,

$$\begin{aligned} \frac{\partial}{\partial t} p(t, x, y) &= \mathcal{L}^{\kappa(y)} p_y(t, x - y) + q(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \partial_t p_z(t - s, x - z) q(s, z, y) dz ds \\ &= \mathcal{L}^{\kappa(y)} p_y(t, x - y) + q(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\kappa(z)} p_z(t - s, x - z) q(s, z, y) dz ds, \end{aligned}$$

while

$$\begin{aligned}
\mathcal{L}^{\kappa(x)}p(t, x, y) &= \mathcal{L}^{\kappa(x)}p_y(t, x - y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\kappa(x)}p_z(t - s, x - z)q(s, z, y)dzds \\
&= \mathcal{L}^{\kappa(x)}p_y(t, x - y) + \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q(s, z, y)dzds \\
&\quad + \int_{\mathbb{R}^d} \mathcal{L}^{\kappa(z)}p_z(t - s, x - z)q(s, z, y)dzds,
\end{aligned}$$

where

$$q_0(t, x, z) = (\mathcal{L}^{\kappa(x)} - \mathcal{L}^{\kappa(z)})p_z(t, x - z).$$

It follows from (3.13) that $q(t, x, y)$ should satisfy

$$q(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q(s, z, y)dzds. \quad (3.15)$$

Thus for the construction and the upper bound heat kernel estimates of $p(t, x, y)$, the main task is to solve $q(t, x, y)$, and to make the above argument rigorous. We use Picard's iteration to solve (3.15). For $n \geq 1$, define

$$q_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q_{n-1}(s, z, y)dzds. \quad (3.16)$$

Then it can be shown that

$$q(t, x, y) := \sum_{n=0}^{\infty} q_n(t, x, y) \quad (3.17)$$

converges absolutely and locally uniformly on $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, $q(t, x, y)$ is jointly continuous in (t, x, y) and has the following upper bound estimate

$$|q(t, x, y)| \leq C \left(\varrho_0^\beta + \varrho_\beta^0 \right) (t, x - y),$$

where

$$\varrho_\gamma^\beta(t, x) := \frac{t^{\gamma/\alpha}(|x|^\beta \wedge 1)}{(t^{1/\alpha} + |x|)^{d+\alpha}}.$$

We then need to address the following issues.

- (i) Show that $p(t, x, y)$ constructed through (3.14) and (3.17) is non-negative, has the property $\int_{\mathbb{R}^d} p(t, x, y)dy = 1$ and satisfies the Chapman-Kolmogorov equation.
- (ii) The kernel $p(t, x, y)$ has the claimed two-sided estimates, and derivative estimates.
- (iii) Uniqueness of $p(t, x, y)$.

This requires detailed studies on the kernel $p_\alpha^\kappa(t, x - y)$ for the symmetric Lévy process with Lévy measure $\frac{\kappa(z)}{|z|^{d+\alpha}}dz$, including its fractional derivative estimates, and its continuous dependence on $\kappa(z)$, which will be outlined in the next two subsections.

3.2 Upper bound estimates

Key observation: For any symmetric function $\kappa(z)$ with $\kappa_0 \leq \kappa(z) \leq \kappa_1$, let $\widehat{\kappa}(z) := \kappa(z) - \frac{\kappa_0}{2}$. Since the Lévy process with Lévy measure $\frac{\kappa(z)}{|z|^{d+\alpha}}dz$ can be decomposed as the independent sum of Lévy processes having respectively Lévy measures $\frac{\widehat{\kappa}(z)}{|z|^{d+\alpha}}dz$ and $\frac{\kappa_0/2}{|z|^{d+\alpha}}dz$, we have

$$p_\alpha^{\kappa(z)}(t, x) = \int_{\mathbb{R}^d} p_\alpha^{\kappa_0/2}(t, x - y) p_\alpha^{\widehat{\kappa}(z)}(t, y) dy.$$

Thus the gradient and fractional derivative estimates on $p_\alpha^{\kappa(z)}(t, x)$ can be obtained from those on $p_\alpha^{\kappa_0/2}(t, x)$. On the other hand, it follows from [17] that there is a constant $c = c(d, \kappa_0, \kappa_1) \geq 1$ so that

$$c^{-1} \varrho_\alpha^0(t, x) \leq p_\alpha^{\kappa(z)}(t, x) \leq c \varrho_\alpha^0(t, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (3.18)$$

First one can establish that for $\gamma_1, \gamma_2, \beta_1, \beta_2 \geq 0$ with $\beta_1 + \gamma_1 > 0$ and $\beta_2 + \gamma_2 > 0$,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \varrho_{\gamma_1}^{\beta_1}(t - s, x - z) \varrho_{\gamma_2}^{\beta_2}(s, z) dz ds \\ & \leq \mathcal{B}\left(\frac{\gamma_1 + \beta_1}{\alpha}, \frac{\gamma_2 + \beta_2}{\alpha}\right) \left(\varrho_{\gamma_1 + \gamma_2 + \beta_1 + \beta_2}^0 + \varrho_{\gamma_1 + \gamma_2 + \beta_2}^{\beta_1} + \varrho_{\gamma_1 + \gamma_2 + \beta_1}^{\beta_2} \right)(t, x), \end{aligned} \quad (3.19)$$

where \mathcal{B} denotes the usual β -function.

Next we establish the continuous dependence of $p_\alpha^{\kappa(z)}(t, y)$ on the symmetric function $\kappa(z)$. Let $\kappa(z)$ and $\widetilde{\kappa}(z)$ be two symmetric functions that are bounded between κ_0 and κ_1 . Then for every $0 < \gamma < \alpha/4$, there is a constant $c > 0$ so that the following estimates hold for all $t \in (0, 1]$ and $x \in \mathbb{R}^d$,

$$|p_\alpha^{\kappa(z)}(t, x) - p_\alpha^{\widetilde{\kappa}(z)}(t, x)| \leq c \|\kappa - \widetilde{\kappa}\|_\infty \varrho_\alpha^0(t, x), \quad (3.20)$$

$$|\nabla_x p_\alpha^{\kappa(z)}(t, x) - \nabla_x p_\alpha^{\widetilde{\kappa}(z)}(t, x)| \leq c \|\kappa - \widetilde{\kappa}\|_\infty t^{-1/\alpha} \varrho_\alpha^0(t, x), \quad (3.21)$$

$$\int_{\mathbb{R}^d} |\delta_{p_\alpha^\kappa}(t, x; z) - \delta_{p_\alpha^{\widetilde{\kappa}}}(t, x; z)| \frac{dz}{|z|^{d+\alpha}} \leq c \|\kappa - \widetilde{\kappa}\|_\infty \varrho_\alpha^0(t, x). \quad (3.22)$$

Here $\|\kappa - \widetilde{\kappa}\|_\infty := \sup_{x \in \mathbb{R}^d} |\kappa(z) - \widetilde{\kappa}(z)|$ and $\delta_f(t, x; z) := f(t, x + z) + f(t, x - z) - 2f(t, x)$. The above estimates are established in [25, Theorem 2.5], but with an extra term on their right hand sides. For example, (3.20) corresponds to [25, (2.30)] where the estimate is

$$|p_\alpha^{\kappa(z)}(t, x) - p_\alpha^{\widetilde{\kappa}(z)}(t, x)| \leq c \|\kappa - \widetilde{\kappa}\|_\infty (\varrho_\alpha^0 + \varrho_{\alpha-\gamma}^\gamma)(t, x). \quad (3.23)$$

We take this opportunity to fill a gap in the proof of [25, (3.20)]. The proof there works only for $|x| \geq t^{1/\alpha}$ and $t \in (0, 1]$, as in this case, by [25, (2.2)],

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} \varrho_0^\gamma(t - s, x - y) \varrho_{\alpha-\gamma}^0(s, y) dy ds & \leq c_1 \varrho_{\alpha-\gamma}^0(t, x) \int_0^t \int_{\mathbb{R}^d} \varrho_0^\gamma(s, y) dy ds \\ & \leq c_2 \varrho_{\alpha-\gamma}^0(t, x) t^{\gamma/\alpha} = c_2 \varrho_\alpha^0(t, x), \end{aligned}$$

which gives (3.23) by line 8 on p.284 of [25]. On the other hand, one deduces by the inverse Fourier transform that

$$\sup_{y \in \mathbb{R}^d} |p_\alpha^{\kappa(z)}(t, y) - p_\alpha^{\widetilde{\kappa}(z)}(t, y)| \leq (2\pi)^d \int_{\mathbb{R}^d} |e^{-t\psi_\kappa(\xi)} - e^{-t\psi_{\widetilde{\kappa}}(\xi)}| d\xi \leq c_3 \|\kappa - \widetilde{\kappa}\|_\infty t^{-d/\alpha}.$$

Thus when $|x| \leq t^{1/\alpha}$, $\left| p_\alpha^{\kappa(z)}(t, x) - p_\alpha^{\tilde{\kappa}(z)}(t, x) \right| \leq c_3 \|\kappa - \tilde{\kappa}\|_\infty t^{-d/\alpha} \leq c_4 \|\kappa - \tilde{\kappa}\|_\infty \varrho_\alpha^0(t, x)$.

In fact, by a slight modification of the original proof given in [25] for (3.23), we can get estimate (3.20). Indeed, by the symmetry of $\mathcal{L}^{\kappa(z)}$ and $\mathcal{L}^{\tilde{\kappa}(z)}$,

$$\begin{aligned}
& p_\alpha^{\kappa(z)}(t, x) - p_\alpha^{\tilde{\kappa}(z)}(t, x) \\
&= \int_0^t \frac{d}{ds} \left(\int_{\mathbb{R}^d} p_\alpha^{\kappa(z)}(s, y) p_\alpha^{\tilde{\kappa}(z)}(t-s, x-y) dy \right) ds \\
&= \int_0^t \left(\int_{\mathbb{R}^d} \left(\mathcal{L}_\alpha^{\kappa(z)} p_\alpha^{\kappa(z)}(s, \cdot)(y) p_\alpha^{\tilde{\kappa}(z)}(t-s, x-y) - p_\alpha^{\kappa(z)}(s, y) \mathcal{L}_\alpha^{\tilde{\kappa}(z)} p_\alpha^{\tilde{\kappa}(z)}(t-s, \cdot)(x-y) \right) dy \right) ds \\
&= \int_0^{t/2} \left(\int_{\mathbb{R}^d} p_\alpha^{\kappa(z)}(s, y) \left(\mathcal{L}_\alpha^{\kappa(z)} - \mathcal{L}_\alpha^{\tilde{\kappa}(z)} \right) p_\alpha^{\tilde{\kappa}(z)}(t-s, \cdot)(x-y) dy \right) ds \\
&\quad + \int_{t/2}^t \left(\int_{\mathbb{R}^d} p_\alpha^{\tilde{\kappa}(z)}(t-s, x-y) \left(\mathcal{L}_\alpha^{\kappa(z)} - \mathcal{L}_\alpha^{\tilde{\kappa}(z)} \right) p_\alpha^{\kappa(z)}(s, \cdot)(y) dy \right) ds.
\end{aligned}$$

Hence by (3.18) and [25, (2.28)],

$$\begin{aligned}
|p_\alpha^{\kappa(z)}(t, x) - p_\alpha^{\tilde{\kappa}(z)}(t, x)| &\leq c \|\kappa - \tilde{\kappa}\|_\infty \int_0^{t/2} \int_{\mathbb{R}^d} \varrho_\alpha^0(s, y) \varrho_0^0(t-s, x-y) dy ds \\
&\quad + c \|\kappa - \tilde{\kappa}\|_\infty \int_{t/2}^t \int_{\mathbb{R}^d} \varrho_0^0(s, y) \varrho_\alpha^0(t-s, x-y) dy ds \\
&\leq \frac{c \|\kappa - \tilde{\kappa}\|_\infty}{t} \int_0^t \int_{\mathbb{R}^d} \varrho_\alpha^0(s, y) \varrho_\alpha^0(t-s, x-y) dy ds \\
&\leq c \|\kappa - \tilde{\kappa}\|_\infty \varrho_\alpha^0(t, x).
\end{aligned}$$

The same proof as that for [25, Theorem 2.5] but using (3.20) instead of (3.23) then gives (3.21)-(3.22).

Since

$$\mathcal{L}^{\kappa(z)} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{\kappa(z)}{|z|^{d+\alpha}} dz = \frac{1}{2} \int_{\mathbb{R}^d} \delta f(x; z) \frac{\kappa(z)}{|z|^{d+\alpha}} dz,$$

estimate (3.22) implies that

$$|\mathcal{L}^{\kappa(z)} p_\alpha^{\kappa(z)}(t, x) - \mathcal{L}^{\tilde{\kappa}(z)} p_\alpha^{\tilde{\kappa}(z)}(t, x)| \leq c \|\kappa - \tilde{\kappa}\|_\infty \varrho_\alpha^0(t, x).$$

From these estimates, one can establish the first part ((i)-(iv)) of the Theorem 3.1 as well as

$$p(t, x, y) \geq ct^{-d/\alpha} \quad \text{for } t \in (0, 1] \text{ and } |x - y| \leq 3t^{1/\alpha}. \quad (3.24)$$

3.3 Lower bound estimates

The upper bound estimates in Theorem 3.1 are established by using analytic method, while the lower bound estimate in Theorem 3.1 are obtained mainly by probabilistic argument.

From (i)-(iv) of Theorem 3.1, we see that $P_t f(x) := \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$ is a Feller semigroup. Hence, it determines a Feller process $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}, (X_t)_{t \geq 0})$.

We first claim the following.

Theorem 3.2 Let $\mathcal{F}_t := \sigma\{X_s, s \leq t\}$. Then for each $x \in \mathbb{R}^d$ and every $f \in C_b^2(\mathbb{R}^d)$, under \mathbb{P}_x ,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds \text{ is an } \mathcal{F}_t\text{-martingale.} \quad (3.25)$$

In other words, \mathbb{P}_x solves the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. Thus \mathbb{P}_x in particular solves the martingale problem for $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$.

Sketch of Proof. For $f \in C_b^2(\mathbb{R}^d)$, define $u(t, x) = f(x) + \int_0^t P_s \mathcal{L}f(x)ds$. Then we have by (3.4) in Theorem 3.1 that

$$\mathcal{L}u(t, x) = \mathcal{L}f(x) + \int_0^t \mathcal{L}P_s \mathcal{L}f(x)ds = \mathcal{L}f(x) + \int_0^t \partial_s(P_s \mathcal{L}f)(x) = P_t \mathcal{L}f(x) = \partial_t u(t, x).$$

Since $P_t f$ also satisfies the equation $\partial_t P_t f = \mathcal{L}(P_t f)$ with $P_0 f = f$, we have

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{L}f(x)ds. \quad (3.26)$$

The desired property (3.25) now follows from (3.26) and the Markov property of X . \square

Theorem 3.2 allows us to derive a Lévy system of X by following an approach from [15]. It is easy to see from (3.25) that $X_t = (X_t^1, \dots, X_t^d)$ is a semi-martingale. For any $f \in C_c^\infty(\mathbb{R}^d)$, we have by Itô's formula that

$$f(X_t) - f(X_0) = \sum_{i=1}^d \int_0^t \partial_i f(X_{s-}) dX_s^i + \sum_{s \leq t} \eta_s(f) + \frac{1}{2} \gamma_t(f), \quad (3.27)$$

where

$$\eta_s(f) = f(X_s) - f(X_{s-}) - \sum_{i=1}^d \partial_i f(X_{s-})(X_s^i - X_{s-}^i) \quad (3.28)$$

and

$$\gamma_t(f) = \sum_{i,j=1}^d \int_0^t \partial_i \partial_j f(X_{s-}) d\langle X^{i,c}, X^{j,c} \rangle_s. \quad (3.29)$$

Here $X^{i,c}$ is the continuous local martingale part of the semimartingale X^i .

Now suppose that A and B are two bounded closed subsets of \mathbb{R}^d having a positive distance from each other. Let $f \in C_c^\infty(\mathbb{R}^d)$ with $f = 0$ on A and $f = 1$ on B . Let M^f be defined as in (3.25). Clearly, $N_t^f := \int_0^t \mathbb{1}_A(X_{s-}) dM_s^f$ is a martingale. Define

$$J(x, y) = k(x, y - x)/|y - x|^{d+\alpha}, \quad (3.30)$$

so \mathcal{L} can be rewritten as

$$\mathcal{L}f(x) = \lim_{\varepsilon \rightarrow 0} \int_{\{|y-x|>\varepsilon\}} (f(y) - f(x)) J(x, y) dy. \quad (3.31)$$

We get by (3.25)–(3.29) and (3.31),

$$\begin{aligned} N_t^f &= \sum_{s \leq t} \mathbb{1}_A(X_{s-})(f(X_s) - f(X_{s-})) - \int_0^t \mathbb{1}_A(X_s) \mathcal{L}f(X_s) ds \\ &= \sum_{s \leq t} \mathbb{1}_A(X_{s-})f(X_s) - \int_0^t \mathbb{1}_A(X_s) \int_{\mathbb{R}^d} f(y) J(X_s, y) dy ds. \end{aligned}$$

By taking a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ with $f_n = 0$ on A , $f_n = 1$ on B and $f_n \downarrow \mathbb{1}_B$, we get that, for any $x \in \mathbb{R}^d$,

$$\sum_{s \leq t} \mathbb{1}_A(X_{s-})\mathbb{1}_B(X_s) - \int_0^t \mathbb{1}_A(X_s) \int_B J(X_s, y) dy ds$$

is a martingale with respect to \mathbb{P}_x . Thus,

$$\mathbb{E}_x \left[\sum_{s \leq t} \mathbb{1}_A(X_{s-})\mathbb{1}_B(X_s) \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} \mathbb{1}_A(X_s)\mathbb{1}_B(y) J(X_s, y) dy ds \right].$$

Using this and a routine measure theoretic argument, we get

$$\mathbb{E}_x \left[\sum_{s \leq t} f(X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} f(X_s, y) J(X_s, y) dy ds \right]$$

for any non-negative measurable function f on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. Finally, following the same arguments as in [17, Lemma 4.7] and [18, Appendix A], we get

Theorem 3.3 *X has a Lévy system (J, t) with J given by (3.3); that is, for any $x \in \mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and (\mathcal{F}_t) -stopping time T ,*

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y) J(X_s, y) dy \right) ds \right]. \quad (3.32)$$

For a set $K \subset \mathbb{R}^d$, denote

$$\sigma_K := \inf\{t \geq 0 : X_t \in K\}, \quad \tau_K := \inf\{t \geq 0 : X_t \notin K\}.$$

Let $B(x, r)$ be the ball with radius r and center x . We need the following lemma (see [2, 17]).

Lemma 3.4 *For each $\gamma \in (0, 1)$, there exists $R_0 > 0$ such that for every $R > R_0$ and $r \in (0, 1)$,*

$$\mathbb{P}_x(\tau_{B(x, Rr)} \leq r^\alpha) \leq \gamma. \quad (3.33)$$

Proof. Without loss of generality, we assume that $x = 0$. Given $f \in C_b^2(\mathbb{R}^d)$ with $f(0) = 0$ and $f(x) = 1$ for $|x| \geq 1$, we set

$$f_r(x) := f(x/r), \quad r > 0.$$

By the definition of f_r , we have

$$\mathbb{P}_0(\tau_{B(0,Rr)} \leq r^\alpha) \leq \mathbb{E}_0 \left[f_{Rr}(X_{\tau_{B(0,Rr)} \wedge r^\alpha}) \right] \stackrel{(3.25)}{=} \mathbb{E}_0 \left(\int_0^{\tau_{B(0,Rr)} \wedge r^\alpha} \mathcal{L} f_{Rr}(X_s) ds \right). \quad (3.34)$$

On the other hand, by the definition of \mathcal{L} , we have for $\lambda > 0$,

$$\begin{aligned} |\mathcal{L} f_{Rr}(x)| &= \frac{1}{2} \left| \int_{\mathbb{R}^d} (f_{Rr}(x+z) + f_{Rr}(x-z) - 2f_{Rr}(x)) \kappa(x, z) |z|^{-d-\alpha} dz \right| \\ &\leq \frac{\kappa_1 \|\nabla^2 f_{Rr}\|_\infty}{2} \int_{|z| \leq \lambda r} |z|^{2-d-\alpha} dz + 2\kappa_1 \|f_{Rr}\|_\infty \int_{|z| \geq \lambda r} |z|^{-d-\alpha} dz \\ &= \kappa_1 \frac{\|\nabla^2 f\|_\infty}{(Rr)^2} \frac{(\lambda r)^{2-\alpha}}{2(2-\alpha)} s_1 + 2\kappa_1 \|f\|_\infty \frac{(\lambda r)^{-\alpha}}{\alpha} s_1 \\ &= \kappa_1 s_1 \left(\frac{\|\nabla^2 f\|_\infty}{R^2} \frac{\lambda^{2-\alpha}}{2(2-\alpha)} + 2\|f\|_\infty \frac{\lambda^{-\alpha}}{\alpha} \right) r^{-\alpha}, \end{aligned}$$

where s_1 is the sphere area of the unit ball. Substituting this into (3.34), we get

$$\mathbb{P}_0(\tau_{B(0,Rr)} \leq r^\alpha) \leq \kappa_1 s_1 \left(\frac{\|\nabla^2 f\|_\infty}{R^2} \frac{\lambda^{2(2-\alpha)}}{2-\alpha} + 2\|f\|_\infty \frac{\lambda^{-\alpha}}{\alpha} \right).$$

Choosing first λ large enough and then R large enough yield the desired estimate. \square

We can now proceed to establish the lower bound heat kernel estimate (3.11). By Lemma 3.4, there is a constant $\lambda \in (0, \frac{1}{2})$ such that for all $t \in (0, 1)$,

$$\mathbb{P}_x(\tau_{B(x, t^{1/\alpha}/2)} > \lambda t) \geq \frac{1}{2}. \quad (3.35)$$

In view of the estimate (3.24), it remains to consider the case that $|x - y| > 3t^{1/\alpha}$. Using (3.35) and the Lévy system of X ,

$$\begin{aligned} &\mathbb{P}_x(X_{\lambda t} \in B(y, t^{1/\alpha})) \\ &\geq \mathbb{P}_x \left(X \text{ hits } B(y, t^{1/\alpha}/2) \text{ before } \lambda t \text{ and then travels less than} \right. \\ &\quad \left. \text{distance } t^{1/\alpha}/2 \text{ for at least } \lambda t \text{ units of time} \right) \\ &\geq \mathbb{P}_x(\sigma_{B(y, t^{1/\alpha}/2)} < \lambda t) \inf_{z \in B(y, t^{1/\alpha}/2)} \mathbb{P}_z(\tau_{B(z, t^{1/\alpha}/2)} > \lambda t) \\ &\geq c_1 \mathbb{P}_x(X_{(\lambda t) \wedge \tau_{B(x, t^{1/\alpha})}} \in B(y, t^{1/\alpha}/2)) \\ &= \mathbb{E}_x \int_0^{(\lambda t) \wedge \tau_{B(x, t^{1/\alpha})}} \int_{B(y, t^{1/\alpha}/2)} J(X_s, u) du ds \\ &\geq c_2 \mathbb{E}_x \left[(\lambda t) \wedge \tau_{B(x, t^{1/\alpha})} \right] \int_{B(y, t^{1/\alpha}/2)} \frac{1}{|x - y|^{d+\alpha}} du \\ &\geq c_3 \frac{t^{(d+\alpha)/\alpha}}{|x - y|^{d+\alpha}}. \end{aligned}$$

Thus

$$\begin{aligned}
p(t, x, y) &\geq \int_{B(y, t^{1/\alpha})} p(\lambda t, x, z) p((1 - \lambda)t, z, y) dz \\
&\geq \mathbb{P}_x(X_{\lambda t} \in B(y, t^{1/\alpha})) \inf_{z \in B(y, t^{1/\alpha})} p((1 - \lambda)t, z, y) \\
&\geq c_4 t^{-d/\alpha} t^{(d+\alpha)/\alpha} \frac{1}{|x - y|^{d+\alpha}} \\
&= \frac{c_4 t}{|x - y|^{d+\alpha}}.
\end{aligned}$$

This proves that

$$p(t, x, y) \geq c \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \quad \text{for every } x, y \in \mathbb{R}^d \text{ and } t \leq 1.$$

3.4 Strong stability

In real world applications and modeling, state-dependent parameter $\kappa(x, z)$ of (3.1) is an approximation of real data. So a natural question is how reliable the conclusion is when using such an approximation. The following strong stability result is recently obtained in [26].

Theorem 3.5 *Suppose $\beta \in (0, \alpha/4]$, and κ and $\tilde{\kappa}$ are two functions satisfying (2.6) and (2.7). Denote the corresponding fundamental solution by $p^\kappa(t, x, y)$ and $p^{\tilde{\kappa}}(t, x, y)$, respectively. Then for every $\gamma \in (0, \beta)$ and $\eta \in (0, 1)$, there exists a constant $C = C(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2, \gamma, \eta) > 0$ so that for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,*

$$|p_\alpha^\kappa(t, x, y) - p_\alpha^{\tilde{\kappa}}(t, x, y)| \leq C \|\kappa - \tilde{\kappa}\|_\infty^{1-\eta} \left(1 + t^{-\gamma/\alpha} (|x - y|^\gamma \wedge 1) \right) \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}. \quad (3.36)$$

Here $\|\kappa - \tilde{\kappa}\|_\infty := \sup_{x, z \in \mathbb{R}^d} |\kappa(x, z) - \tilde{\kappa}(x, z)|$.

Observe that by (3.5) and (3.11), the term $\frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}$ in (3.36) is comparable to $p_\alpha^\kappa(t, x, y)$ and to $p_\alpha^{\tilde{\kappa}}(t, x, y)$. So the error bound (3.36) is also a relative error bound, which is good even in the region when $|x - y|$ is large.

Let $\{P_t^\kappa; t \geq 0\}$ and $\{P_t^{\tilde{\kappa}}; t \geq 0\}$ be the semigroups generated by \mathcal{L}^κ and $\mathcal{L}^{\tilde{\kappa}}$, respectively. For $p \geq 1$, denote by $\|P_t^\kappa - P_t^{\tilde{\kappa}}\|_{p,p}$ the operator norm of $P_t^\kappa - P_t^{\tilde{\kappa}}$ in Banach space $L^p(\mathbb{R}^d; dx)$.

Corollary 3.6 *Suppose $\beta \in (0, \alpha/4]$, and κ and $\tilde{\kappa}$ are two functions satisfying (2.6) and (2.7). Then for every $\gamma \in (0, \beta)$ and $\eta \in (0, 1)$, there exists a constant $C = C(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2, \gamma, \eta) > 0$ so that for every $p \geq 1$ and $t \in (0, 1]$,*

$$\|P_t^\kappa - P_t^{\tilde{\kappa}}\|_{p,p} \leq C t^{-\gamma/\alpha} \|\kappa - \tilde{\kappa}\|_\infty^{1-\eta}. \quad (3.37)$$

Theorem 3.5 is derived by estimating each $|q_n^\kappa(t, x, y) - q_n^{\tilde{\kappa}}(t, x, y)|$ for $q_n^\kappa(t, x, y)$ and $q_n^{\tilde{\kappa}}(t, x, y)$ of (3.16). Corollary 3.6 is a direct consequence of Theorem 3.5.

For uniformly elliptic divergence form operators \mathcal{L} and $\tilde{\mathcal{L}}$ on \mathbb{R}^d , pointwise estimate on $|p(t, x, y) - \tilde{p}(t, x, y)|$ and the L^p -operator norm estimates on $P_t - \tilde{P}_t$ are obtained in Chen, Hu, Qian and Zheng [14] in terms of the local L^2 -distance between the diffusion matrix of \mathcal{L} and $\tilde{\mathcal{L}}$. Recently, Bass and Ren [4] obtained strong stability result for symmetric α -stable-like non-local operators of (3.3), with error bound expressed in terms of the L^q -norm on the function $c(x) := \sup_{y \in \mathbb{R}^d} |c(x, y) - \tilde{c}(x, y)|$.

3.5 Applications to SDE driven by stable processes

Suppose that $\sigma(x) = (\sigma_{ij}(x))_{1 \leq i, j \leq d}$ is a bounded continuous $d \times d$ -matrix-valued function on \mathbb{R}^d that is non-degenerate at every $x \in \mathbb{R}^d$, and Y is a (rotationally) symmetric α -stable process on \mathbb{R}^d for some $0 < \alpha < 2$. It is shown in Bass and Chen [1, Theorem 7.1] that for every $x \in \mathbb{R}^d$, SDE

$$dX_t = \sigma(X_{t-})dY_t, \quad X_0 = x, \quad (3.38)$$

has a unique weak solution. (Although in [1] it is assumed $d \geq 2$, the results there are valid for $d = 1$ as well.) The family of these weak solutions forms a strong Markov process $\{X, \mathbb{P}_x, x \in \mathbb{R}^d\}$. Using Itô's formula, one deduces (see the display above (7.2) in [1]) that X has generator

$$\mathcal{L}f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x + \sigma(x)y) - f(x)) \frac{\mathcal{A}(d, -\alpha)}{|y|^{d+\alpha}} dy. \quad (3.39)$$

A change of variable formula $z = \sigma(x)y$ yields

$$\mathcal{L}f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x + z) - f(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz, \quad (3.40)$$

where

$$\kappa(x, z) = \frac{\mathcal{A}(d, -\alpha)}{|\det \sigma(x)|} \left(\frac{|z|}{|\sigma(x)^{-1}z|} \right)^{d+\alpha}. \quad (3.41)$$

Here $\det(\sigma(x))$ is the determinant of the matrix $\sigma(x)$ and $\sigma(x)^{-1}$ is the inverse of $\sigma(x)$. As an application of Theorem 3.1, we have

Corollary 3.7 ([1, Corollary 1.2]) *Suppose that $\sigma(x) = (\sigma_{ij}(x))$ is a $d \times d$ matrix-valued function on \mathbb{R}^d such that there are positive constants $\lambda_0, \lambda_1, \lambda_2$ and $\beta \in (0, 1)$ so that*

$$\lambda_0 \mathbf{I}_{d \times d} \leq \sigma(x) \leq \lambda_1 \mathbf{I}_{d \times d} \quad \text{for every } x \in \mathbb{R}^d, \quad (3.42)$$

and

$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq \lambda_2 |x - y|^\beta \quad \text{for } 1 \leq i, j \leq d. \quad (3.43)$$

Then the strong Markov process X formed by the unique weak solution to SDE (3.38) has a jointly continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure on \mathbb{R}^d , and there is a constant $C \geq 1$ that depends only on $(d, \alpha, \beta, \lambda_0, \lambda_1)$ so that

$$C^{-1} \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}} \leq p(t, x, y) \leq C \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}$$

for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$. Moreover, $p(t, x, y)$ enjoys all the properties stated in the conclusions of Theorem 3.1 with $\kappa_0 = \mathcal{A}(d, -\alpha) \lambda_0^{d+\alpha} \lambda_1^{-d}$, $\kappa_1 = \mathcal{A}(d, -\alpha) \lambda_0^{-d} \lambda_1^{d+\alpha}$ and $\kappa_2 = \kappa_2(d, \lambda_0, \lambda_1, \lambda_2)$.

The following strong stability result for SDE (3.38) is a direct consequence of Corollary 3.6 and (3.41).

Corollary 3.8 *Suppose that $\sigma(x) = (\sigma_{ij}(x))$ and $\tilde{\sigma}(x) = (\tilde{\sigma}_{ij}(x))$ are $d \times d$ matrix-valued functions on \mathbb{R}^d satisfying conditions (3.42) and (3.43). Let $p(t, x, y)$ and $\tilde{p}(t, x, y)$ be the transition density functions of the corresponding strong Markov processes X and \tilde{X} that solve SDE (3.38), respectively. Then for every $\gamma \in (0, \beta)$ and $\eta \in (0, 1)$, there exists a constant $C = C(d, \alpha, \beta, \lambda_0, \lambda_1, \lambda_2, \gamma, \eta) > 0$ so that for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,*

$$|p(t, x, y) - \tilde{p}(t, x, y)| \leq C \|\sigma - \tilde{\sigma}\|_\infty^{1-\eta} \left(1 + t^{-\gamma/\alpha}(|x - y|^\gamma \wedge 1)\right) \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}, \quad (3.44)$$

where $\|\sigma - \tilde{\sigma}\|_\infty^{1-\eta} := \sum_{i,j=1}^d \sup_{x,y \in \mathbb{R}^d} |\sigma_{ij}(x) - \tilde{\sigma}_{ij}(x)|$.

4 Diffusion with jumps

In this section, we consider non-local operators that have both elliptic differential operator part and pure non-local part:

$$\mathcal{L}f(x) := \mathcal{L}^a f(x) + b \cdot \nabla f(x) + \mathcal{L}^\kappa f(x), \quad (4.1)$$

where

$$\begin{aligned} \mathcal{L}^a f(x) &:= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \partial_{ij}^2 f(x), & b \cdot \nabla f(x) &:= \sum_{i=1}^d b_i(x) \partial_i f(x), \\ \mathcal{L}^\kappa f(x) &:= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \mathbb{1}_{\{|z| \leq 1\}} z \cdot \nabla f(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz. \end{aligned}$$

Here $a(x) := (a_{ij}(x))_{1 \leq i,j \leq d}$ is a $d \times d$ -symmetric matrix-valued measurable function on \mathbb{R}^d , $b(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\kappa(x, z) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ are measurable functions, and $\alpha \in (0, 2)$.

For convenience, we assume $d \geq 2$. Throughout this section, we impose the following assumptions on a and κ :

(H^a) There are $c_1 > 0$ and $\beta \in (0, 1)$ such that for any $x, y \in \mathbb{R}^d$,

$$|a(x) - a(y)| \leq c_1 |x - y|^\beta, \quad (4.2)$$

and for some $c_2 \geq 1$,

$$c_2^{-1} \mathbf{I}_{d \times d} \leq a(x) \leq c_2 \mathbf{I}_{d \times d}. \quad (4.3)$$

(H^κ) $\kappa(x, z)$ is a bounded measurable function and if $\alpha = 1$, we require

$$\int_{r < |z| \leq R} \kappa(x, z) |z|^{-d-1} dz = 0 \quad \text{for any } 0 < r < R < \infty. \quad (4.4)$$

Note that when $\kappa(x, z)$ is a positive constant function,

$$\mathcal{L} = \mathcal{L}^a + b \cdot \nabla + c\Delta^{\alpha/2}$$

for some constant $c > 0$. A function f defined on \mathbb{R}^d is said to be in Kato class \mathbb{K}_2 if $f \in L_{loc}^1(\mathbb{R}^d)$ and

$$\lim_{\delta \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^\delta \int_{\mathbb{R}^d} \frac{t^{1/2} |f(y)|}{(t^{1/2} + |x - y|)^{d+2}} dy dt = 0. \quad (4.5)$$

Let $q(t, x, y)$ be the fundamental solution of $\{\mathcal{L}^a; t \geq 0\}$; see Theorem 4.3 below for more information. Since \mathcal{L} can be viewed as a perturbation of \mathcal{L}^a by $\mathcal{L}^{b, \kappa} := b \cdot \nabla + \mathcal{L}^\kappa$, heuristically the fundamental solution (or heat kernel) $p(t, x, y)$ of \mathcal{L} should satisfy the following Duhamel's formula: for all $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p(t, x, y) = q(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p(r, x, z) \mathcal{L}^{b, \kappa} q(t - r, \cdot, y)(z) dz dr \quad (4.6)$$

or

$$p(t, x, y) = q(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q(r, x, z) \mathcal{L}^{b, \kappa} p(t - r, \cdot, y)(z) dz dr. \quad (4.7)$$

The following is a special case of the main results in [13], where the corresponding results are also obtained for time-inhomogeneous operators.

Theorem 4.1 ([13, Theorem 1.1]) *Let $\alpha \in (0, 2)$. Under (\mathbf{H}^a) , (\mathbf{H}^κ) and $b \in \mathbb{K}_2$, there is a unique continuous function $p(t, x; y)$ that satisfies (4.6), and*

- (i) *(Upper-bound estimate) For any $T > 0$, there exist constants $C_0, \lambda_0 > 0$ such that for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$|p(t, x, y)| \leq C_0 \left(t^{-d/2} e^{-\lambda_0 |x-y|^2/t} + \frac{\|\kappa\|_\infty t}{(t^{1/2} + |x - y|)^{d+\alpha}} \right). \quad (4.8)$$

- (ii) *(C-K equation) For all $s, t > 0$ and $x, y \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} p(s, x, y) p(t, y, z) dy = p(s + t, x, z). \quad (4.9)$$

- (iii) *(Gradient estimate) For any $T > 0$, there exist constants $C_1, \lambda_1 > 0$ such that for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$|\nabla_x p(t, x, y)| \leq C_1 t^{-1/2} \left(t^{-d/2} e^{-\lambda_1 |x-y|^2/t} + \frac{\|\kappa\|_\infty t}{(t^{1/2} + |x - y|)^{d+\alpha}} \right). \quad (4.10)$$

- (iv) *(Conservativeness) For any $t > 0$ and $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} p(t, x, y) dy = 1$.*

- (v) *(Generator) Define $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$. Then for any $f \in C_b^2(\mathbb{R}^d)$, we have*

$$P_t f(x) - f(x) = \int_0^t P_s \mathcal{L} f(x) ds. \quad (4.11)$$

(vi) (Continuity) For any bounded and uniformly continuous function f , $\lim_{t \rightarrow 0} \|P_t f - f\|_\infty = 0$.

Define $m_\kappa = \inf_{x \in \mathbb{R}^d} \operatorname{ess\,inf}_{z \in \mathbb{R}^d} k(x, z)$.

Theorem 4.2 ([13, Theorem 1.3]) *If κ is a bounded function satisfying (\mathbf{H}^κ) and that for each $x \in \mathbb{R}^d$,*

$$\kappa(x, z) \geq 0 \quad \text{for a.e. } z \in \mathbb{R}^d, \quad (4.12)$$

then $p(t, x, y) \geq 0$. Furthermore, if $m_\kappa > 0$, then for any $T > 0$, there are constants $C_1, \lambda_2 > 0$ such that for any $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$p(t, x, y) \geq C_1 \left(t^{-d/2} e^{-\lambda_2 |x-y|^2/t} + \frac{m_\kappa t}{(t^{1/2} + |x-y|)^{d+\alpha}} \right). \quad (4.13)$$

We have by Theorems 4.1 and 4.2 that when $\kappa \geq 0$, then there is a conservative Feller process $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ having $p(t, x, y)$ as its transition density function with respect to the Lebesgue measure. It follows from (4.11) that X is a solution to the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$.

When a is the identity matrix, $b = 0$ and $\kappa(x, z)$ is a positive constant, $\mathcal{L} = \Delta + c\Delta^{\alpha/2}$ for some positive constant $c > 0$. In this case, the corresponding Markov process X is a symmetric Lévy process that is the sum of a Brownian motion W and an independent rotationally symmetric α -stable process Y . Thus the heat kernel $p(t, x, y)$ for \mathcal{L} is the convolution of the transition density function of W and Y . In this case, its two-sided bounds can be obtained through a direct calculation. Indeed such a computation is carried out in Song and Vondraček [43].

Symmetric diffusions with jumps corresponding to symmetric non-local operators on \mathbb{R}^d with variable coefficients of the the following form have been studied in [19]:

$$\mathcal{L}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial f(x)}{\partial x_j} \right) + \lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} (f(y) - f(x)) \frac{c(x, y)}{|x-y|^{d+\alpha}} dy, \quad (4.14)$$

where $a(x) := (a_{ij}(x))_{1 \leq i, j \leq d}$ is a $d \times d$ -symmetric matrix-valued measurable function on \mathbb{R}^d , $c(x, y)$ is a symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that is bounded between two positive constants, and $\alpha \in (0, 2)$. Clearly, when $a(x)$ is the identity matrix and $c(x, y)$ is a positive constant, the above non-local operator is $\Delta + c_0 \Delta^{\alpha/2}$ for some $c_0 > 0$. Among other things, it is established in Chen and Kumagai [19] that the symmetric non-local operator \mathcal{L} of (4.14) has a jointly Hölder continuous heat kernel $p(t, x, y)$ and there are positive constants c_i , $1 \leq i \leq 4$ so that

$$\begin{aligned} & c_1 \left(t^{-d/2} \wedge t^{-d/\alpha} \right) \wedge \left(t^{-d/2} e^{-c_2 |x-y|^2/t} + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \\ & \leq p(t, x, y) \leq c_3 \left(t^{-d/2} \wedge t^{-d/\alpha} \right) \wedge \left(t^{-d/4} e^{-c_4 |x-y|^2/t} + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \end{aligned} \quad (4.15)$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$. It is easy to see that for each fixed $T > 0$, the two-sided estimates (4.15) on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ is equivalent to

$$\tilde{c}_1 \left(t^{-d/2} e^{-c_2 |x-y|^2/t} + \frac{t}{(t^{1/2} + |x-y|)^{d+\alpha}} \right) \leq p(t, x, y) \leq \tilde{c}_3 \left(t^{-d/2} e^{-c_4 |x-y|^2/t} + \frac{t}{(t^{1/2} + |x-y|)^{d+\alpha}} \right).$$

When a is the identity matrix and $b = 0$, the results in Theorems 4.1 and 4.2 have been obtained recently in [47] for $\kappa(x, z)$ that is symmetric in z .

4.1 Approach

The approach in [13] is to treat \mathcal{L} as \mathcal{L}^a under lower order perturbation $b \cdot \nabla + \mathcal{L}^\kappa$, and thus one can construct the fundamental solution for \mathcal{L} from that of \mathcal{L}^a through Duhamel's formula.

The following result is essentially known in literature; see [27] (see also [13, Theorem 2.3]).

Theorem 4.3 *Under (\mathbf{H}^a) , there exists a nonnegative continuous function $q(t, x, y)$, called the fundamental solution or heat kernel of \mathcal{L}^a , with the following properties:*

- (i) *(Two-sided estimates) For any $T > 0$, there exist constants $C, \lambda > 0$ such that for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$C^{-1}t^{-d/2}e^{-\lambda^{-1}|x-y|^2/t} \leq q(t, x, y) \leq Ct^{-d/2}e^{-\lambda|x-y|^2/t}. \quad (4.16)$$

- (ii) *(Gradient estimate) For $j = 1, 2$ and $T > 0$, there exist constants $C, \lambda > 0$ such that for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$|\nabla_x^j q(t, x, y)| \leq Ct^{-(d+j)/2}e^{-\lambda|x-y|^2/t}. \quad (4.17)$$

- (iii) *(Hölder estimate in y) For $j = 0, 1$, $\eta \in (0, \beta)$ and $T > 0$, there exist constants $C, \lambda > 0$ such that for $t \in (0, T]$, $x, y, z \in \mathbb{R}^d$,*

$$|\nabla_x^j q(t, x, y) - \nabla_x^j q(t, x, z)| \leq C|y - z|^\eta t^{-(d+j+\eta)/2} \left(e^{-\lambda|x-y|^2/t} + e^{-\lambda|x-z|^2/t} \right). \quad (4.18)$$

Moreover, for bounded measurable $f : \mathbb{R}^d \rightarrow \mathbb{R}$, let $Q_t f(x) := \int_{\mathbb{R}^d} q(t, x, y) f(y) dy$. We have

- (iv) *(Continuity) For any bounded and uniformly continuous function f , $\lim_{t \rightarrow 0} \|Q_t f - f\|_\infty = 0$.*

- (v) *(C-K equation) For all $0 \leq t < r < s < \infty$, $Q_t Q_s = Q_{t+s}$.*

- (vi) *(Conservativeness) For all $0 \leq t < s < \infty$, $Q_t 1 = 1$.*

- (vii) *(Generator) For any $f \in C_b^2(\mathbb{R}^d)$, we have*

$$Q_t f(x) - f(x) = \int_0^t Q_s \mathcal{L}^a f(x) dr = \int_0^t \mathcal{L}^a Q_s f(x) ds.$$

As mentioned earlier, it is expected that the fundamental solution $p(t, x, y)$ of \mathcal{L} should satisfy Duhamel's formula (4.6). We construct $p(t, x, y)$ recursively. Let $p_0(t, x, y) = q(t, x, y)$, and define for $n \geq 1$,

$$p_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} p_{n-1}(t-s, x, z) \mathcal{L}^{b, \kappa} q(s, \cdot, y)(z) dz ds.$$

Using Theorem 4.3, one can show that $p_n(t, x, y)$ is well defined and that $\sum_{n=0}^\infty p_n(t, x, y)$ converges locally uniformly to some function $p(t, x, y)$, and that $p(t, x, y)$ is the unique solution stated in

Theorem 4.1. The positivity (4.12) of Theorem 4.2 can be established by using Hille-Yosida-Ray theorem and Courr ge's first theorem.

The Gaussian part in the lower bound estimate on $p(t, x, y)$ in Theorem 4.2 is obtained from the near diagonal lower bound estimate on $p(t, x, y)$ and a chaining ball argument, while the pure jump part in the lower bound estimate on $p(t, x, y)$ is obtained by using a probabilistic argument through the L vy system, similar to that in Section 3.

4.2 Application to SDE

Let $\sigma(x)$ be a $d \times d$ -matrix valued function on \mathbb{R}^d that is uniformly elliptic and bounded, and each entry σ_{ij} is β -H lder continuous on \mathbb{R}^d , $b \in \mathbb{K}_2$ and $\tilde{\sigma}$ a bounded $d \times d$ -matrix valued measurable function on \mathbb{R}^d . Suppose X solves the following stochastic differential equation:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + \tilde{\sigma}(X_{t-})dY_t,$$

where W is a Brownian motion on \mathbb{R}^d and Y is a rotationally symmetric α -stable process on \mathbb{R}^d . By It 's formula, the infinitesimal generator \mathcal{L} of X is of the form $\mathcal{L}^a + b \cdot \nabla + \mathcal{L}^\kappa$ with $a(x) = \sigma(x)\sigma(x)^*$ and

$$\kappa(x, z) = \frac{\mathcal{A}(d, -\alpha)}{|\det \tilde{\sigma}(x)|} \left(\frac{|z|}{|\tilde{\sigma}(x)^{-1}z|} \right)^{d+\alpha}.$$

So by Theorems 4.1 and 4.2, X has a transition density function $p(t, x, y)$ satisfying the properties there. If in addition, $\tilde{\sigma}$ is uniformly elliptic, then for any $T > 0$,

$$c_1 \left(t^{-d/2} e^{-\lambda_1 |x-y|^2/t} + \frac{t}{(t^{1/2} + |x-y|)^{d+\alpha}} \right) \leq p(t, x, y) \leq c_2 \left(t^{-d/2} e^{-\lambda_2 |x-y|^2/t} + \frac{t}{(t^{1/2} + |x-y|)^{d+\alpha}} \right)$$

for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$.

5 Other related work

In this section, we briefly mention some other recent work on heat kernels of non-symmetric non-local operators.

Using a perturbation argument, Bogdan and Jakubowski [7] constructed a *particular* heat kernel (also called fundamental solution) $q^b(t, x, y)$ for operator $\mathcal{L}^b := \Delta^{\alpha/2} + b \cdot \nabla$ on \mathbb{R}^d , where $d \geq 1$, $\alpha \in (1, 2)$ and b is a function on \mathbb{R}^d that is in a suitable Kato class. It is based on the following heuristics: $q^b(t, x, y)$ of \mathcal{L}^b can be related to the fundamental solution $p(t, x, y)$ of $\mathcal{L}^0 = \Delta^{\alpha/2}$, which is the transition density of the rotationally symmetric α -stable process Y , by the following Duhamel's formula:

$$q^b(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^b(s, x, z) b(z) \cdot \nabla_z p(t-s, z, y) dz ds. \quad (5.1)$$

Applying the above formula recursively, one expects that

$$q^b(t, x, y) := \sum_{k=0}^{\infty} q_k^b(t, x, y) \quad (5.2)$$

is a fundamental solution for \mathcal{L}^b , where $q_0^b(t, x, y) := p(t, x, y)$ and for $k \geq 1$,

$$q_k^b(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_{k-1}^b(s, x, z) b(z) \cdot \nabla_z p(t-s, z, y) dz.$$

It is shown in [7] that the series in (5.2) converges absolutely and, for every $T > 0$, such defined $q^b(t, x, y)$ is a conservative transition density function and is comparable to $p(t, x, y)$ on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Recall that $p(t, x, y)$ has two-sided estimate (2.5). In [22], Chen and Wang showed that the Markov process X_t having $q^b(t, x, y)$ as its transition density function is the unique solution to the martingale problem $(\mathcal{L}^b, C_b^2(\mathbb{R}^d))$; moreover, it is the unique weak solution to the following stochastic differential equation:

$$dX_t = dY_t + b(X_t)dt,$$

where Y_t is the rotationally symmetric α -stable process on \mathbb{R}^d . Dirichlet heat kernel estimate for \mathcal{L}^b in a bounded $C^{1,1}$ open set has been obtained in [15]. In [31, 32], Kim and Song extended results in [7, 16] to $\Delta^{\alpha/2} + \mu \cdot \nabla$, where $\mu = (\mu_1, \dots, \mu_d)$ are signed measures in suitable Kato class. These work can be regarded as heat kernels for fractional Laplacian under gradient perturbation. Heat kernel estimates for relativistic stable processes and for mixed Brownian motions and stable processes with drifts have recently been studied in [23] and [12], respectively. See [11] for drift perturbation of subordinate Brownian motion of pure jump type and its heat kernel estimate. While in [49], Xie and Zhang considered the critical operator $\mathcal{L}^b := a\Delta^{1/2} + b \cdot \nabla$, where for some $0 < c_0 < c_1$, $a : \mathbb{R}^d \rightarrow [c_0, c_1]$ and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are two Hölder continuous functions. They established two-sided estimates for the heat kernel of \mathcal{L}^b by using Levi's method as described in Subsection 3.1.

In the same spirit, Wang and Zhang in [45] considered more general fractional diffusion operators over a complete Riemannian manifold perturbed by a time-dependent gradient term, and showed two-sided estimates and gradient estimate of the heat kernel. More precisely, let M be a d -dimensional connected complete Riemannian manifold with Riemannian distance ρ . Let Δ^M be the Laplace-Beltrami operator. Suppose that the heat kernel $p(t, x, y)$ of Δ^M with respect to the Riemannian volume dx exists and has the following two-sided estimates:

$$c_1 t^{-d/2} e^{-c_2 \rho(x, y)^2/t} \leq p(t, x, y) \leq c_3 t^{-d/2} e^{-c_4 \rho(x, y)^2/t}, \quad t > 0, x, y \in M, \quad (5.3)$$

and gradient estimate

$$|\nabla_x p(t, x, y)| \leq c_5 t^{-(d+1)/2} e^{-c_4 \rho(x, y)^2/t}, \quad (5.4)$$

where ∇_x denotes the covariant derivative. Let P_t be the corresponding semigroup, that is,

$$P_t f(x) := \int_M p(t, x, y) f(y) dx, \quad f \in C_b(M).$$

For $0 < \alpha < 2$, consider the $(\alpha/2)$ -stable subordination of P_t

$$P_t^{(\alpha)} := \int_0^\infty P_s \mu_t^{(\alpha/2)}(ds), \quad t \geq 0,$$

where $\mu_t^{(\alpha/2)}$ is a probability measure on $[0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-\lambda s} \mu_t^{(\alpha/2)}(ds) = e^{-t\lambda^{\alpha/2}}, \quad \lambda \geq 0.$$

Then $P_t^{(\alpha)}$ is a C_0 -contraction semigroup on $C_b(M)$. Let $\mathcal{L}^{(\alpha)}$ be the infinitesimal generator of $P_t^{(\alpha)}$. In [45], Wang and Zhang considered the following operator

$$\mathcal{L}_{b,c}^{(\alpha)} f(t, x) := \mathcal{L}^{(\alpha)} f(x) + \langle b(t, x), \nabla_x f(x) \rangle + c(t, x) f(x), \quad f \in C_b^2(M),$$

where $b : \mathbb{R}_+ \times M \rightarrow TM$ and $c : \mathbb{R}_+ \times M \rightarrow \mathbb{R}$ are measurable. For $\alpha \in (0, 2)$, one says that a measurable function $f : \mathbb{R}_+ \times M \rightarrow \mathbb{R}$ belongs to Kato's class \mathbb{K}_α if

$$\lim_{\varepsilon \rightarrow 0} \sup_{(t,x) \in [0,\infty) \times M} \varepsilon^{1/\alpha} \int_0^\varepsilon \int_M \frac{s^{1-1/\alpha} (\varepsilon - s)^{-1/\alpha} |f(t \pm s, y)|}{(s^{1/\alpha} + \rho(x, y))^{d+\alpha}} dy ds = 0.$$

Notice that when $\alpha = 2$ and f is time-independent, \mathbb{K}_α is the same as in (4.5).

The following result is shown in [45].

Theorem 5.1 *Assume (5.3), (5.4) and $\alpha \in (1, 2)$. If $|b|, c \in \mathbb{K}_\alpha$, then there is a unique continuous function $p_{b,c}^{(\alpha)}(t, x; s, y)$ having the following properties:*

(i) *(Two-sided estimates) There is a constant $c_1 > 0$ such that for all $t - s \in (0, 1], x, y \in M$,*

$$c_1^{-1} \frac{t - s}{((t - s)^{1/\alpha} + \rho(x, y))^{d+\alpha}} \leq p_{b,c}^{(\alpha)}(t, x; s, y) \leq c_1 \frac{t - s}{((t - s)^{1/\alpha} + \rho(x, y))^{d+\alpha}}.$$

(ii) *(Gradient estimate) There is a constant $c_2 > 0$ such that for all $t - s \in (0, 1], x, y \in M$,*

$$|\nabla_x p_{b,c}^{(\alpha)}(t, x; s, y)| \leq c_2 \frac{(t - s)^{1-1/\alpha}}{((t - s)^{1/\alpha} + \rho(x, y))^{d+\alpha}}.$$

(iii) *(C-K equation) For any $0 \leq s < r < t$ and $x, y \in M$,*

$$p_{b,c}^{(\alpha)}(t, x; s, y) = \int_M p_{b,c}^{(\alpha)}(t, x; r, z) p_{b,c}^{(\alpha)}(r, z; s, y) dz.$$

(iv) *(Generator) If $b \in C([0, \infty); L_{loc}^1(M, dx; TM))$ and $c \in C([0, \infty); L_{loc}^1(M, dx; \mathbb{R}))$, then for any $\varphi, \psi \in C_0^2(M)$,*

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_M \psi(P_{t,s}^{b,c} \varphi - \varphi) dx = \int_M \psi \mathcal{L}_{b,c}^{(\alpha)}(s, \cdot) \varphi dx, \quad s \geq 0,$$

$$\text{where } P_{t,s}^{b,c} \varphi := \int_M p_{b,c}^{(\alpha)}(t, \cdot; s, y) \varphi(y) dy.$$

The above results indicate that, under suitable Kato class condition, heat kernel estimates are stable under gradient perturbation.

In [20], Chen and Wang studied heat kernels for fractional Laplacian under non-local perturbation of high intensity; that is, heat kernels for

$$\mathcal{L}^\kappa f(x) = \Delta^{\alpha/2} f(x) + \mathcal{S}^\kappa f(x), \quad f \in C_b^2(\mathbb{R}^d), \quad (5.5)$$

where

$$\mathcal{S}^\kappa f(x) := \mathcal{A}(d, -\beta) \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}}) \frac{\kappa(x, z)}{|z|^{d+\beta}} dz \quad (5.6)$$

for some $0 < \beta < \alpha < 2$ and a real-valued bounded function $\kappa(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\kappa(x, z) = \kappa(x, -z) \quad \text{for every } x, z \in \mathbb{R}^d.$$

Uniqueness and existence of fundamental solution $q^\kappa(t, x, y)$ is established in [20]. The approach is also a perturbation argument by viewing $\mathcal{L}^\kappa = \Delta^{\alpha/2} + \mathcal{S}^\kappa$ as a lower order perturbation of $\mathcal{L}^0 = \Delta^{\alpha/2}$ by \mathcal{S}^κ . So heuristically, the fundamental solution (or heat kernel) $q^\kappa(t, x, y)$ of \mathcal{L}^κ should satisfy the following Duhamel's formula:

$$q^\kappa(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^\kappa(t-s, x, z) \mathcal{S}_z^\kappa p(s, z, y) dz ds \quad (5.7)$$

for $t > 0$ and $x, y \in \mathbb{R}^d$. Here the notation $\mathcal{S}_z^\kappa p(s, z, y)$ means that the non-local operator \mathcal{S}^κ is applied to the function $z \mapsto p(s, z, y)$. Similar notation will also be used for other operators, for example, $\Delta_z^{\alpha/2}$. Applying (5.7) recursively, it is reasonable to conjecture that $\sum_{n=0}^\infty q_n^\kappa(t, x, y)$, if convergent, is a solution to (5.7), where $q_0^\kappa(t, x, y) := p(t, x, y)$ and

$$q_n^\kappa(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_{n-1}^\kappa(t-s, x, z) \mathcal{S}_z^\kappa p(s, z, y) dz ds \quad \text{for } n \geq 1. \quad (5.8)$$

The hard part is the estimates on $\mathcal{S}_z^\kappa p(s, z, y)$ and on each $q_n^\kappa(t, x, y)$. In contrast to the gradient perturbation case, the fundamental solution to the non-local perturbation (5.5) does not need to be positive, and when the kernel is positive, it does not need to be comparable to $p(t, x, y)$. One can rewrite \mathcal{L}^κ of (5.5) as follows:

$$\mathcal{L}^\kappa f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) j^\kappa(x, z) dz,$$

where

$$j^\kappa(x, z) = \frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} \left(1 + \frac{\mathcal{A}(d, -\beta)}{\mathcal{A}(d, -\alpha)} \kappa(x, z) |z|^{\alpha-\beta} \right). \quad (5.9)$$

It is shown in [20] that the fundamental solution kernel $q^\kappa \geq 0$ if $j^\kappa(x, z) \geq 0$; that is, if

$$\kappa(x, z) \geq -\frac{\mathcal{A}(d, -\alpha)}{\mathcal{A}(d, -\beta)} |z|^{\beta-\alpha} \quad \text{for a.e. } z \in \mathbb{R}^d. \quad (5.10)$$

When $\kappa(x, z)$ is continuous in x , the above condition is also necessary for the non-negativity of $q^\kappa(t, x, y)$. Under condition (5.10), various sharp heat kernel estimates have been obtained in [20].

In particular, it is shown in [20] that if there are constants $0 < c_1 \leq c_2$ so that

$$\frac{c_1}{|z|^{d+\alpha}} \leq j^\kappa(x, z) \leq \frac{c_2}{|z|^{d+\alpha}} \quad \text{for } x, z \in \mathbb{R}^d,$$

then for every $T > 0$, $q^\kappa(t, x, y) \asymp p(t, x, y)$ on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Dirichlet heat kernel estimates for \mathcal{L}^κ of (5.5) has recently been studied in Chen and Yang [24].

In a subsequent work [47], Wang studied fundamental solution for $\Delta + \mathcal{S}^\kappa$ and its two-sided heat kernel estimates. In [16], Chen, Kim and Song established stability of heat kernel estimates under (local and non-local) Feynman-Kac transforms for a class of jump processes; see also C. Wang [44] on a related work. Very recently, stability of heat kernel estimates for diffusions with jumps (both symmetric and non-symmetric) under Feynman-Kac transform has been studied in Chen and Wang [21]. On the other hand, by employing the strategy and road map from Chen and Zhang [25] as outlined in Section 3 of this paper, Kim, Song and Vondracek [33] has extended Theorem 3.1 to more general non-local operator \mathcal{L} of (3.1) with $\frac{1}{|z|^{d+\alpha}}$ being replaced by the density of Lévy measure of certain subordinate Brownian motions. In a recent work [9], X. Chen, Z.-Q. Chen and J. Wang have used Levi's freezing coefficient method to obtain upper and lower bound estimates for heat kernels of the following type of non-local operators of variable order:

$$\mathcal{L}f(x) := \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \right) \frac{\kappa(x, z)}{|z|^{d+\alpha(x)}} dz, \quad f \in C_c^2(\mathbb{R}^d),$$

where $\alpha(x)$ is a Hölder continuous function on \mathbb{R}^d such that

$$0 < \alpha_1 \leq \alpha(x) \leq \alpha_2 < 2 \quad \text{for all } x \in \mathbb{R}^d,$$

and $\kappa(x, z)$ satisfies conditions (2.6)-(2.7).

In this survey, we mainly concentrate on the quantitative estimates of the heat kernels of non-symmetric nonlocal operators. For derivative formula of the heat kernel associated with stochastic differential equations with jumps, we refer the interested reader to [50, 48, 46]. For other results on the existence and smoothness of heat kernels or fundamental solutions for non-symmetric jump processes or non-local operators under Hörmander's type conditions, see [41, 37] for the studies of linear Ornstein-Uhlenbeck processes with jumps, and [51, 52, 53] and the references therein for the studies of general stochastic differential equations with jumps. We will not survey these results since the arguments in the above references are mainly based on the Malliavin calculus and thus belong to another topic.

Acknowledgement. We thank the referee for helpful comments, in particular for pointing out a gap in the proof of (2.30) in [25].

References

- [1] R. F. Bass and Z.-Q. Chen, Systems of equations driven by stable processes. *Probab. Theory Relat. Fields* **134** (2006), 175–214.
- [2] R. F. Bass and D. A. Levin, Harnack inequalities for jump processes. *Potential Anal.* **17** (2002), 375–388.
- [3] R. F. Bass and D. A. Levin, Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.* **354** (2002), 2933–2953.

- [4] R. F. Bass and H. Ren, Meyers inequality and strong stability for stable-like operators. *J. Funct. Anal.* **265** (2013), 28-48.
- [5] J. Bertoin. *Lévy Processes*. Cambridge University Press, 1996.
- [6] R. M. Blumenthal and R. K. Gettoor, Some theorems on stable processes. *Trans. Amer. Math. Soc.* **95** (1960), 263-273.
- [7] K. Bogdan and T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operator. *Commun. Math. Phys.* **271**, (2007)179-198.
- [8] L. A. Caffarelli, S. Salsa and Luis Silvestre. Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian. *Invent. Math.* **171**(1) (2008) 425–461.
- [9] X. Chen, Z.-Q. Chen and J. Wang, Heat kernel for non-local operators with variable order. Preprint.
- [10] Z.-Q. Chen, Symmetric jump processes and their heat kernel estimates. *Sci. China Ser. A.* **52** (2009), 1423-1445.
- [11] Z.-Q. Chen and X. Dou, Drift perturbation of subordinate Brownian motions with Gaussian component. *Sci. China Math.* **59** (2016), 239-260.
- [12] Z.-Q. Chen and E. Hu, Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ under gradient perturbation. *Stochastic Process. Appl.* **125** (2015), 2603-2642.
- [13] Z.-Q. Chen, E. Hu, L. Xie and X. Zhang, Heat kernels for non-symmetric diffusions operators with jumps. Preprint 2016.
- [14] Z.-Q. Chen, Y. Hu, Z. Qian and W. Zheng, Stability and approximations of symmetric diffusion semigroups and kernels. *J. Funct. Anal.* **152** (1998), 255-280.
- [15] Z.-Q. Chen, P. Kim and R. Song, Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation. *Ann. Probab.* **40** (2012), 2483-2538.
- [16] Z.-Q. Chen, P. Kim and R. Song, Stability of Dirichlet heat kernel estimates for non-local operators under Feynman-Kac perturbation. *Trans. Amer. Math. Soc.* **367** (2015), 5237-5270.
- [17] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d -sets. *Stochastic Process. Appl.* **108** (2003), 27-62.
- [18] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Related Fields* **140** (2008), 277-317.
- [19] Z.-Q. Chen and T. Kumagai, A priori Hölder estimate, parabolic Harnack inequality and heat kernel estimates for diffusions with jumps. *Revista Matemática Iberoamericana* **26** (2010), 551-589.
- [20] Z.-Q. Chen and J.-M. Wang, Perturbation by non-local operators. To appear in *Ann. Inst. Henri Poincaré Probab. Statist.*
- [21] Z.-Q. Chen and Lidan Wang, Stability of heat kernel estimates for diffusions with jumps under non-local Feynman-Kac perturbations. arXiv:1702.04489 [math.PR]
- [22] Z.-Q. Chen and Longmin Wang, Uniqueness of stable processes with drifts. *Proc. Amer. Math. Soc.* **144** (2016), 2661-2675.

- [23] Z.-Q. Chen and Longman Wang, Heat kernel estimates for relativistic stable processes with singular drifts. Preprint.
- [24] Z.-Q. Chen and T. Yang, Dirichlet heat kernel estimates for fractional Laplacian under non-local perturbation. Preprint.
- [25] Z.-Q. Chen and X. Zhang, Heat kernels and analyticity of non-symmetric jump diffusion semi-groups. *Probab. Theory Relat. Fields* **165** (2016), 267-312.
- [26] Z.-Q. Chen and X. Zhang, Strong stability of heat kernels of non-symmetric stable-like operators. To appear in *Stochastic Analysis and Related Topics, A Festschrift in honor of Rodrigo Banuelos*. Birkhauser.
- [27] A. Friedman, *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, N.J., 1975.
- [28] A. Grigor'yan, J. Hu and K.-S. Lau, Heat kernels on metric measure spaces. *Geometry and analysis of fractals*, 147-207, Springer Proc. Math. Stat., 88, Springer, Heidelberg, 2014.
- [29] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α -Stable Processes*. Dekker, 1994.
- [30] N. Jacob, *Pseudo Differential Operators and Markov Processes*. Vo. I-III. Imperial College Press, London, 2001/2002/2005.
- [31] P. Kim and R. Song, Stable process with singular drift. *Stochastic Process Appl.* **124** (2014), 2479–2516.
- [32] P. Kim and R. Song, Dirichlet heat kernel estimates for stable processes with singular drift in unbounded $C^{1,1}$ open sets. *Potential Anal.* **41** (2014), 555-581.
- [33] P. Kim, R. Song and Z. Vondracek, Heat kernels of non-symmetric jump processes: beyond the stable case. arXiv:1606.02005v2 [math.PR]
- [34] J. Klafter, M. F. Shlesinger and G. Zumofen, Beyond Brownian motion. *Physics Today*, **49** (1996), 33–39.
- [35] V. Kolokoltsov, Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc.* **80** (2000), 725–768.
- [36] T. Komatsu, On the martingale problem for generators of stable processes with perturbations. *Osaka J. Math.* **21** (1984), 113-132.
- [37] A. Kulik: Conditions for existence and smoothness of the distribution density for Ornstein-Uhlenbeck processes with Lévy noises. *Theory Probab. Math. Statist.* no.79(2009), 23-38.
- [38] A. Matacz, Financial modeling and option theory with the truncated Lévy process. *Int. J. Theor. Appl. Finance* **3**(1) (2000), 143–160.
- [39] G. Pólya, On the zeros of an integral function represented by Fourier's integral. *Messenger of Math.* **52** (1923), 185-188.
- [40] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York-London, 1994.
- [41] E. Priola and J. Zabczyk, Densities for Ornstein-Uhlenbeck processes with jumps. *Bull. Lond. Math. Soc.* **41** (2009), 41-50.
- [42] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* **60** (2007), 67-112.

- [43] R. Song and Z. Vondraček, Parabolic Harnack inequality for the mixture of Brownian motion and stable process. *Tohoku Math. J.* **59** (2007), 1–19.
- [44] C. Wang, On estimates of the density of Feynman-Kac semigroups of α -stable-like processes. *J. Math. Anal. Appl.* **348** (2008), 938–970.
- [45] F.-Y. Wang and X. Zhang, Heat kernel for fractional diffusion operators with perturbations. *Forum Mathematicum* **27** (2015), 973–994.
- [46] F.-Y. Wang, L. Xu and X. Zhang X., Gradient estimates for SDEs Driven by Multiplicative Lévy Noise. *J. Funct. Anal.* **269** (2015), 3195–3219.
- [47] J.-M. Wang, Laplacian perturbed by non-local operators. *Math. Z.* **279** (2015), 521–556.
- [48] L. Wang, L. Xie and X. Zhang, Derivative formulae for SDEs driven by multiplicative α -stable-like processes, *Stochastic Process Appl.* **125** (2015), 867–885.
- [49] L. Xie and X. Zhang, Heat kernel estimates for critical fractional diffusion operator. *Studia Math.* **224** (2014), 221–263.
- [50] X. Zhang, Derivative formula and gradient estimate for SDEs driven by α -stable processes. *Stochastic Process Appl.* **123** (2013), 1213–1228.
- [51] X. Zhang, Fundamental solution of kinetic Fokker-Planck operator with anisotropic nonlocal dissipativity. *SIAM J. Math. Anal.* **46**, No. 3 (2014), 2254–2280.
- [52] X. Zhang, Densities for SDEs driven by degenerate α -stable processes. *Ann. Probab.* **42** (2014), 1885–1910.
- [53] X. Zhang, Fundamental solutions of nonlocal Hörmander’s operators. *Commun. Math. Stat.* **4** (2016), 359–402.
- [54] X. Zhang, Fundamental solutions of nonlocal Hörmander’s operators II. *Ann. Probab.*, DOI: 10.1214/16-AOP1102.

Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA

E-mail: zqchen@uw.edu

Xicheng Zhang

School of Mathematics and Statistics, Wuhan University, Hubei 430072, P. R. China

E-mail: XichengZhang@gmail.com